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Braiding a link with a fixed closed braid

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ABSTRACT

We show that every oriented link diagram with a closed braid diagram as a sublink diagram can be deformed into a closed braid diagram by a deformation keeping the sublink diagram and, under a mild condition, the number of Seifert circles fixed. As an application, we give an upper bound for the braid index of the link obtained by reversing the orientation of its sublink by using only the information of an original link.

1. Introduction

Throughout this paper, all links and link diagrams are oriented. Alexander [1] showed that every link can be deformed into a closed braid. Yamada [9] provided a braiding deformation for every link diagram, which preserves the number of Seifert circles. Lambropoulou and Rourke [5] showed that every link with a sublink which is in the form of a closed braid can be deformed into a closed braid keeping the sublink fixed. We note that their deformation does not give any information on the number of Seifert circles. In this paper, we show that every link diagram such that a sublink diagram is in the form of a closed braid diagram and the remaining sublink diagram is in a braid box, can be deformed into a closed braid diagram by a deformation keeping the sublink diagram and the number of Seifert circles of the link diagram fixed (see Theorem 1). Since every link with a sublink in the closed braid form is deformed into a link with such a diagram by a deformation keeping the sublink fixed, our result gives an alternative proof of the Lambropoulou–Rourke result in [5].

Let L be a link which is in the form of a closed braid, and L' a link obtained from L by reversing the orientation of a sublink of L. As an application of Theorem 1, we give an upper bound for the braid index of L' by using only the information of L (Theorem 2). Our upper bound also gives an upper bound for the span of the HOMFLY polynomial $P_{L'}(v,z)$ in the variable v. This result is something new, because a difference of the span between $P_L(v,z)$ and $P_{L'}(v,z)$ is not well understood (cf. [4]), although the spans of the Jones and Kauffman polynomials are unchanged (cf. [6]).

2. Preliminaries

Set $I^3 = \{(x,y,z) \in \mathbb{R}^3 \mid x,y,z \in [0,1]\}$. An *n-braid* is the set of n disjoint strands in I^3 between n points in $\{1/2\} \times [0,1] \times \{1\}$ and n points in $\{1/2\} \times [0,1] \times \{0\}$ such that the strands are monotone with respect to the z-coordinate. We assume that all strands of any braid have the top-to-bottom directions. An *n-braid diagram* of an n-braid is the diagram in $I^2 = \{(x,y,z) \in I^3 \mid x = 0\}$ obtained by projecting the n-braid to I^2. A *closed n-braid* $\hat{b}$ is a link obtained from an n-braid b by connecting the end-points with n parallel strands in $\mathbb{R}^3 \setminus I^3$ (see Fig. 1). We denote by $\natural b$ the number of strands of the braid b.

Let L be a link which is in the form of a closed braid $\hat{b}$, and L_1 a sublink of L. A *subbraid* b_1 of b is a subset of b such that $b_1 = L_1 \cap I^3$. The *braid index* of a link L, denoted by $\mathrm{braid}(L)$, is the minimal number of strands of any braid whose closure is equivalent to L. We obtain simple loops by smoothing all crossings of a link diagram D, where the smoothing

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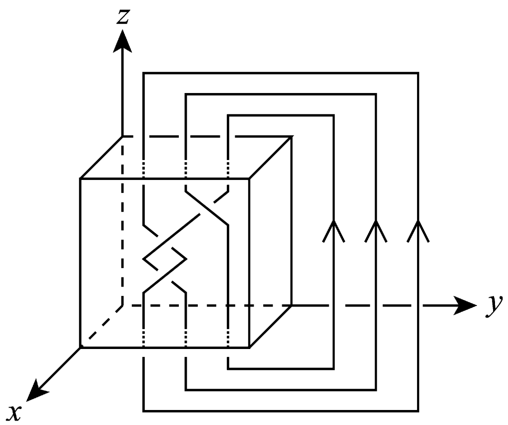

Fig. 1.

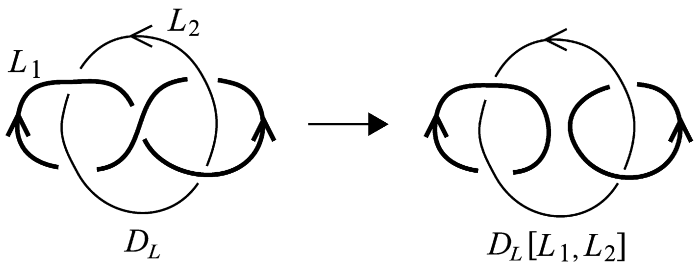

Fig. 2.

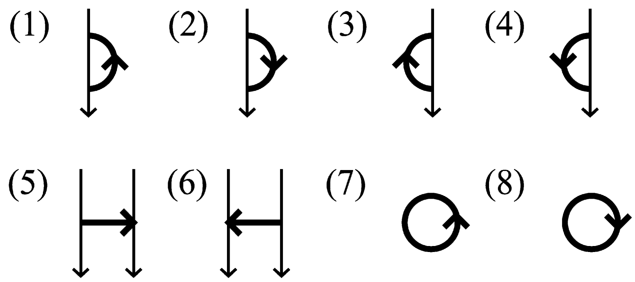

Fig. 3.

operation is applied along the orientations of arcs. A *Seifert circle* of D is a loop obtained in this way. Let $s(D)$ be the number of Seifert circles of D. We denote a link L by $L_1 \cup L_2$ if L_1 and L_2 are sublinks of L and $L_2 = L \setminus L_1$. We call a component of L_i $(i = 1, 2)$ an L_i*-component*. For a diagram D_L of L, an (L_1, L_2)*-crossing* of D_L is a crossing between an L_1-component and an L_2-component. We denote by $D_L[L_1, L_2]$ the link diagram obtained from D_L by smoothing all crossings except for the (L_1, L_2)-crossings (see Fig. 2).

3. A braiding deformation that keeps a sublink diagram fixed

Let $L = L_1 \cup L_2$ be a link such that L_1 is in I^3, and that L_2 is in the form of a closed braid. Let D_L be the diagram of L. An L_1*-arc* of $D_L[L_1, L_2]$ is a segment of an L_1-component in the y–z plane between any two consecutive (L_1, L_2)-crossings. An L_1-arc of $D_L[L_1, L_2]$ is one of the eight types (1)–(8) represented in Fig. 3, where we represent an L_1-arc of D_L by a thick arc and omit over/under information at a crossing. We call an L_1-arc of the type (1) or (3) (resp. (2) or (4)) an *upward* (resp. *downward*) L_1*-arc* of $D_L[L_1, L_2]$. We call an L_1-arc of the type (7) or (8) an L_1*-loop* of $D_L[L_1, L_2]$. We denote by $\#_u^{L_1}(D_L)$ (resp. $\#_d^{L_1}(D_L)$) the number of upward (resp. downward) L_1-arcs of $D_L[L_1, L_2]$, and by $\#_l^{L_1}(D_L)$ the number of L_1-loops of $D_L[L_1, L_2]$.

In the next theorem, we show that D_L can be isotopically deformed into a closed braid diagram while keeping the image of the sublink L_2 fixed so that the number of Seifert circles of D_L is preserved.

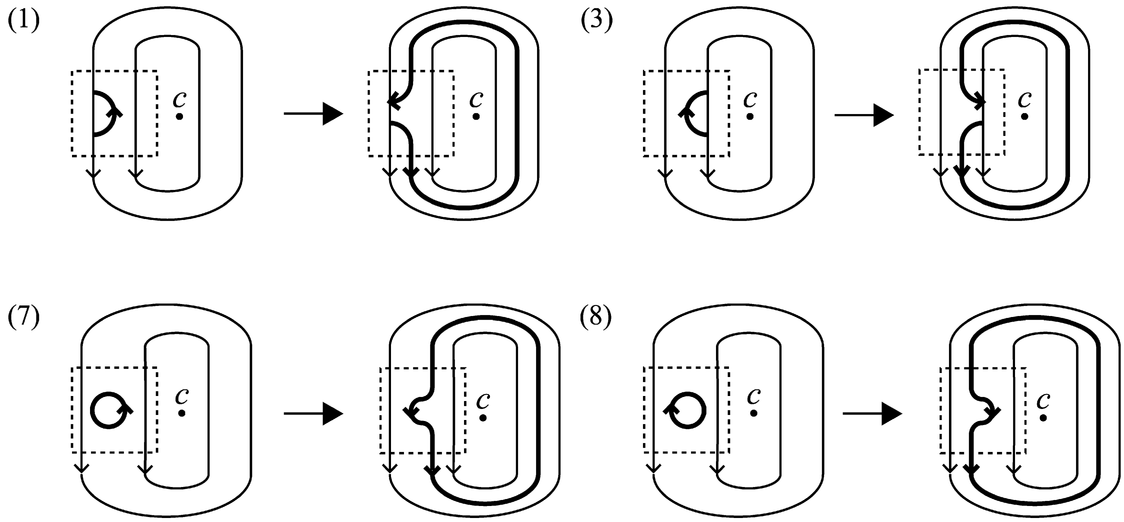

Fig. 4.

Theorem 1. *Let $L = L_1 \cup L_2$ be a link such that L_1 is in I^3 and that L_2 is in the form of a closed braid $\hat{b}_2$. Then we can deform the diagram D_L of L into a closed braid diagram D'_L while keeping the image of the sublink L_2 fixed so that the number of Seifert circles is preserved*:

$$s(D_L) = s(D'_L) = \#_u^{L_1}(D_L) + \#_l^{L_1}(D_L) + \natural b_2.$$

Proof. We give a braiding deformation for D_L. Let c be a point in the y–z plane such that the closed braid diagram of L_2 is around c. For an arc r of D_L which corresponds to an upward L_1-arc (resp. L_1-loop) of $D_L[L_1, L_2]$, we deform r into an arc r' which is around the point c as shown in Fig. 4(1), (3) (resp. (7), (8)) (cf. [9]). By applying this deformation at all arcs in D_L which correspond to upward L_1-arcs and L_1-loops of $D_L[L_1, L_2]$, we obtain a link diagram D'_L. Since all arcs of $D'_L \cap I^2$ have no maximal point with respect to the z-coordinate, D'_L is a closed braid diagram. This deformation preserves the number of Seifert circles, because we have

$$s(D_L) = s(D_L[L_1, L_2]) = \#_u^{L_1}(D_L) + \#_l^{L_1}(D_L) + \natural b_2 = s(D'_L).$$

This completes the proof. □

4. An upper bound for the braid index of a closed braid

Let $L = L_1 \cup L_2$ be a link which is in the form of a closed braid. Let $D_L = D_1 \cup D_2$ be the diagram of $L_1 \cup L_2$. We denote by $(-D_1) \cup D_2$ the diagram obtained from D_L by reversing the orientation of D_1, and denote by $(-L_1) \cup L_2$ the oriented link represented by $(-D_1) \cup D_2$. An (L_1, L_2)-*full twist* of D_L is an n-full twist tangle diagram between an L_1-arc and an L_2-arc for some $n \neq 0$. Let T_i $(i = 1, 2, \ldots, m)$ be a list of the mutually disjoint (L_1, L_2)-full twists in D_L. Let $t(D_1 \cup D_2) = \sum_{i=1}^{m}(t(T_i) - 1)$, where $t(T_i)$ is half the number of crossings of T_i. In the next theorem, we give an upper bound for the braid index of $(-L_1) \cup L_2$ by using only the information of the diagram of $L_1 \cup L_2$.

Theorem 2. *Let $L = L_1 \cup L_2$ be a link which is in the form of a closed braid $\hat{b} = \hat{b}_1 \cup \hat{b}_2$. Let $D_L = D_1 \cup D_2$ be the diagram of $L = L_1 \cup L_2$. If the y-values of the end-points of b_1 are greater than those of b_2, then we have*

$$\mathrm{braid}\big((-L_1) \cup L_2\big) \leqslant \#_d^{L_1}(D_L) + \#_l^{L_1}(D_L) + \natural b_2 - t(D_1 \cup D_2),$$

where we assume that L_1 is in I^3 (see Fig. 5).

Proof. By Theorem 1, we have

$$\begin{aligned} s\big((-D_1) \cup D_2\big) &= \#_u^{L_1}\big((-D_1) \cup D_2\big) + \#_l^{L_1}\big((-D_1) \cup D_2\big) + \natural b_2 \\ &= \#_d^{L_1}(D_1 \cup D_2) + \#_l^{L_1}(D_1 \cup D_2) + \natural b_2. \end{aligned}$$

Let T' be an anti-parallel full twist of $(-D_1) \cup D_2$ obtained from an (L_1, L_2)-full twist T of $D_1 \cup D_2$ by reversing the orientation of D_1. We obtain the diagram D' from $(-D_1) \cup D_2$ by deforming T' as shown in Fig. 6.

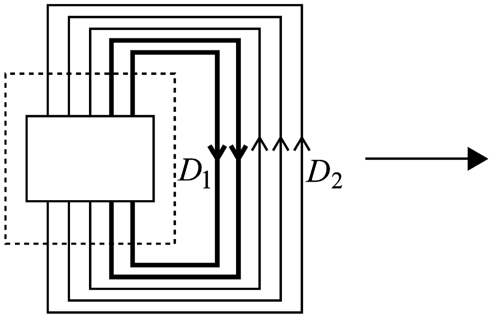

Fig. 5.

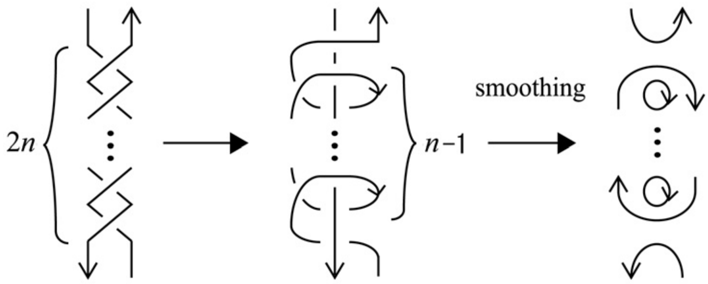

Fig. 6.

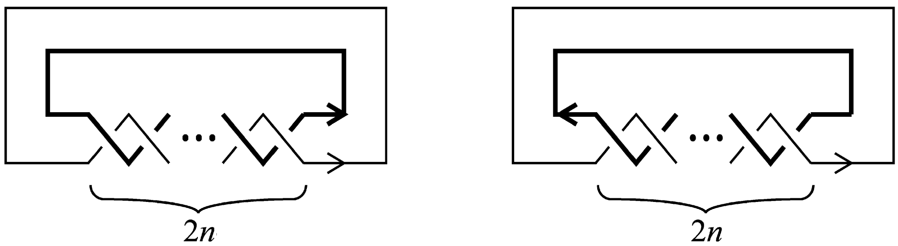

Fig. 7.

Then we have the equality

$$s(D') = s\big((-D_1) \cup D_2\big) - \big(t(T) - 1\big).$$

By applying this deformation at all (L_1, L_2)-full twists of $D_1 \cup D_2$, we have the inequality

$$\mathrm{braid}\big((-L_1) \cup L_2\big) \leqslant \#_d^{L_1}(D_L) + \#_l^{L_1}(D_L) + \natural b_2 - t(D_1 \cup D_2). \qquad \square$$

Let $P_L(v, z)$ be the HOMFLY polynomial of a link L [3,8]. Franks and Williams [2], and Morton [7] independently showed that

$$\frac{1}{2}\big(\mathrm{maxdeg}_v\, P_L(v, z) - \mathrm{mindeg}_v\, P_L(v, z)\big) + 1 \leqslant \mathrm{braid}(L),$$

where $\mathrm{maxdeg}_v\, P_L(v, z)$ (resp. $\mathrm{mindeg}_v\, P_L(v, z)$) is the maximal (resp. minimal) degree of $P_L(v, z)$ in the variable v.

Example 3. (1) Let $D_{2,2n}$ be the closed braid diagram of the $(2, 2n)$-torus link $T_{2,2n}$ as shown in Fig. 7. Then we have $\mathrm{braid}(T_{2,2n}) = 2$. Let $T'_{2,2n}$ be the link obtained from $T_{2,2n}$ by reversing the orientation of one component. By Theorem 2, we have

$$\mathrm{braid}\big(T'_{2,2n}\big) \leqslant 2n - (n - 1) = n + 1.$$

On the other hand, the lower bound for $\mathrm{braid}(T'_{2,2n})$ is given by

$$\frac{1}{2}\big(\mathrm{maxdeg}_v\, P_{T'_{2,2n}}(v, z) - \mathrm{mindeg}_v\, P_{T'_{2,2n}}(v, z)\big) + 1 = n + 1.$$

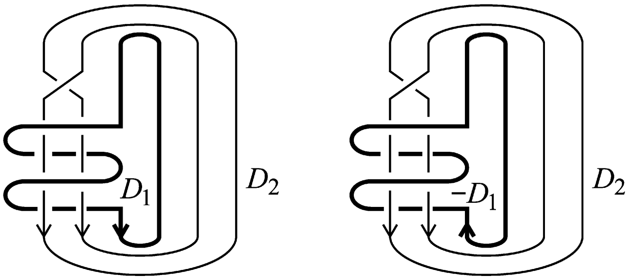

Fig. 8.

Thus, we have

$$\mathrm{braid}(T'_{2,2n}) = n + 1.$$

Then the difference between the braid indices of $T_{2,2n}$ and $T'_{2,2n}$ is $n - 1$.

(2) Let $D_1 \cup D_2$ be a closed braid diagram of a link (see Fig. 8). Then we have $L_1 \cup L_2$ braid$(L_1 \cup L_2) = 3$. By Theorem 2, we have braid$((-L_1) \cup L_2) \leqslant 5$. However this inequality is not sharp because we actually have braid$((-L_1) \cup L_2) = 4$.

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